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Magnetic flux penetration into a non-uniform Josephson junction

V V Bryksin, A V Goltsev, S N Dorogovtsev, Yu I Kuzmin and A N Samukhin
Ioffe Physico-Technical Institute, 194021 St Petersburg, Russia

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Abstract. The extreme profile of a magnetic field penetrating into a non-uniform Josephson junction is computed in the case of periodically arranged pinning centres by use of a numerical solution of a chain of the sine–Gordon equations with boundary conditions at the pinning centres. It is found that the extreme profile obtained is approximately described by the Bean critical-state theory. At sufficiently small spacings we find no stochastic regime for a Josephson junction of a finite length. This shows two possibilities: either the front of the profile is extremely extended or, at magnetic fields larger than H_{c1}, only a uniform vortex distribution is realized.

1. Introduction

When increasing a magnetic field above the low critical magnetic field H_{c1}, vortices penetrate into a Josephson junction (see, e.g., Barone and Paterno 1982, and many others). If the junction is non-uniform, then after a fast exponential relaxation a non-uniform magnetic field distribution is formed. The distribution starts to relax to a more or less uniform distribution which ensures the free-energy minimum. Since the system overcomes many potential barriers, this slow relaxation process is non-exponential.

Such phenomena are well known for hard superconductors of the second type (Anderson 1962, Bean 1962, 1964, Anderson and Kim 1964, de Gennes 1966). This non-uniform state, which is formed after the initial relaxation process has finished, is called 'critical'. In the papers by Anderson (1962), Bean (1962, 1964) and Anderson and Kim (1964) a magnetic induction profile ($B(r)$) has been determined phenomenologically by use of the condition that demands equality of a local current to the critical current J_c at each point of the profile. The function $J_c(H)$ may be found by means of general considerations.

Recently (Bryksin *et al* 1990a, b) we have developed a microscopic theory and have obtained the critical profiles $B(x)$ and $J_c(H)$ for Josephson lattices and specifically for the non-uniform Josephson junction. To determine the critical magnetic profile we have supposed that in the critical state the vortex density along the Josephson junction has maximum possible jumps when crossing pinning centres. Because a phase correlation at different pinning centres was neglected, the vortex density jumps have been found independently at each pinning centre. That is why the problem under consideration becomes local (Bryksin *et al* 1990a, b).

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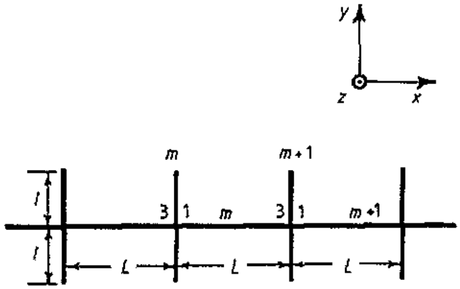

Figure 1. Non-uniform Josephson junction (x–z plane) with regularly arranged defects. L is the spacing, and $2l$ is the length of cross junctions (y–z plane) which act as the defects.

Below we shall present a computer calculation of critical (or extreme) profiles of the vortex density $n(x)$ (or the induction $B(x) \equiv \overline{H(x)}$) in a linear non-uniform Josephson junction. We shall take into account the phase correlation at different pinning centres, computing a chain of the sine–Gordon equations related by boundary conditions. As a result, two regimes are observed. If the distance between pinning centres is large enough, a number of solutions with non-uniform vortex density is found. Among these solutions, a solution with the extreme profile can be found (see crosses in figure 2). The profile is placed under a corresponding profile found by Bryksin *et al* (1990a, b) without taking into account the phase correlation between pinning centres that occurs in the framework of the local approach.

In the case of a short distance between pinning centres which corresponds to the second regime we can find no solution which differs significantly from a periodic one at any rate for a Josephson junction of a finite length with periodically arranged pinning centres. Moreover, no transition to stochastic behaviour is observed (see figure 3). Below we shall discuss magnetic field penetration in the latter case.

2. Model, computation and results

Let us consider a long Josephson junction crossed by short Josephson junctions of length $2l$ and spacing L (figure 1) (Bryksin *et al* 1989a, b). Such defects are equivalent to microresistors (Kivshar and Malomed 1989) or cavities embedded in a Josephson junction. Systems of this type may be obtained practically (Serpuchenko and Ustinov 1987).

Each piece of the long Josephson junction between two neighbouring defects is described by the sine–Gordon equation

$$\delta^2\, \partial^2\vartheta/\partial x^2 = \sin\vartheta \tag{1}$$

where $\vartheta(x)$ is the phase difference and δ is the Josephson length. The derivatives of $\vartheta(x)$ with respect to x on the left and right of the pinning centre with the index m are equal to each other because of the magnetic field continuity (Bryksin *et al* 1990c), i.e.

$$\partial\vartheta_m^{(3)}/\partial x = \partial\vartheta_m^{(1)}/\partial x \equiv \vartheta'_m. \tag{2}$$

The related jump of the phase difference is

$$\vartheta_m^{(1)} - \vartheta_m^{(3)} = 2l\vartheta'_m. \tag{3}$$

Here $\vartheta_m^{(1)}$ and $\vartheta_m^{(3)}$ are the phase differences on the right and left of the mth pinning centre, respectively.

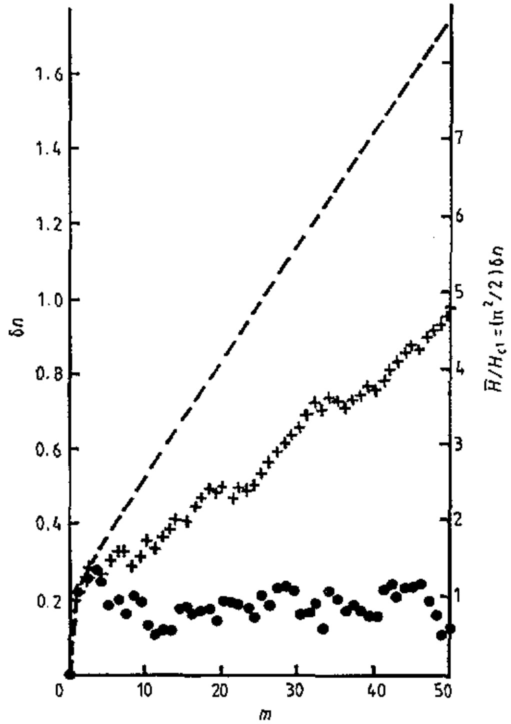

Figure 2. Profile of the vortex density n_m in the junction with parameters $L/\delta = 100$, $l/\delta = 0.1$ where the applied external magnetic field is shown on the right-hand side: +, extreme profile with the initial condition $x_0/\delta = -0.782\,504\,38$: ●, a profile with $x_0/\delta = -0.7835$; ---, the extreme profile calculated with the local approach (Bryksin *et al* 1990a, b). On the left of $m = 0$, i.e. at $-L < x < 0$ the phase difference $\vartheta(x)$ of the Josephson junction decreases in accordance with equation (4). $\bar{H}_m$ is the averaged magnetic field.

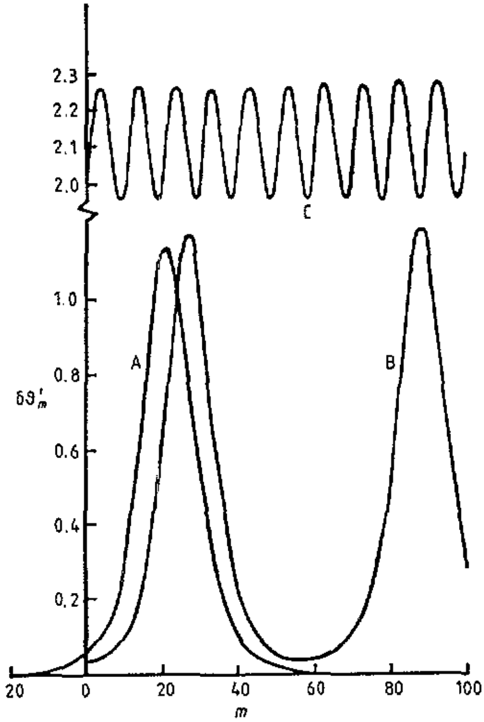

Figure 3. Magnetic field $H_m = \vartheta_m' \Phi_0 / 2\pi d$ at pinning centres versus the site index m for different initial conditions at $m = 0$: curve A, one-vortex solution $(\vartheta_0^{(3)} = 0.1;\ \vartheta_0'\delta = 0.061\,092\,9)$; curve B, almost periodic solution with $\vartheta_0^{(3)} = 0.026\,951\,4$, $\vartheta_0'\delta = 0.026\,906$; curve C almost periodic solution with $\vartheta_0^{(3)} = 2$, $\vartheta_0'\delta = 0.5$.

Let the external magnetic field be applied from the right and the magnetic field penetrates to the left (figure 2). We shall find an extreme profile which is related to a metastable state which has the greatest thermodynamic potential at the applied field. Let us fix the position of the profile front on the assumption that $n(x < 0) = 0$ where $n(x)$ is the vortex density and the coordinate $x = 0$ corresponds to $m = 0$.

First we consider the case of large spacing: $L \gg \delta$. Therefore one supposes that the solution decreasing deep into the system for $-L < x < 0$ has the form

$$\vartheta(x) = 4\tan^{-1}\{\exp[(x - x_0)/\delta]\}. \tag{4}$$

The parameter x_0/δ determines the phase difference $\vartheta_0^{(3)} \equiv \vartheta(0-0)$ and the derivative ϑ_0'. Thus the initial conditions at site $m = 0$ (or at the front) are

$$\vartheta^{(1)} = 4\tan^{-1}[\exp(-x_0/\delta)] + 2l\vartheta'(0)$$

$$\vartheta'(0) = 2/\cosh x_0.$$

This allows one to obtain a solution on the right of $m = x = 0$. Solutions may be very chaotic. Changing x_0, we look for a solution which at certain $m = N$ gives a maximum magnetic field H_N and a maximum vortex density n_N. In this case the extremely steep profile is realized and the minimum number of vortices penetrate into the system through the surface. It should be noted that we construct the solution starting from the front of the magnetic field. If we started from the surface at a fixed ϑ', we would face the problem of determination of a boundary phase ϑ which related to the extreme solution decreasing deep into the system, i.e. $\vartheta(-\infty) = 0$ and $\vartheta(-\infty) = 2\pi k$ where k is integer number.

For a certain x_0 the solution of the considered problem is determined by the following procedure. Using (3) and (4), one obtains $\vartheta_0^{(1)}$, $\vartheta_0^{(3)}$ and ϑ_0'. Then by use of the well known general solutions of the sine–Gordon equation (see, e.g., Barone and Paterno 1982) we can find $\vartheta_1^{(3)}$ and ϑ_1' at the next pinning centre and so on. Therefore the phase correlation between different pinning centres is exactly taken into account.

We obtain that the vortex density $n_N(x_0)$, the flux $\Phi_N(x_0) = (\Phi_0/2\pi)\vartheta_N(x_0)$ which penetrates into the system and the corresponding free energy $\mathcal{F}(x_0)$ at an external magnetic field are not smooth functions of the parameters x_0 and change considerably when x_0 changes insignificantly (see the two solutions in figure 2 shown by crosses and full circles for two values close to x_0). This is why it is difficult to determine the upper local minimum of $\mathcal{F}$ by means of this procedure. Nevertheless with an increasing number of attempts we can approach the extreme profile.

Using the procedure with $L/\delta = 100$, $l/\delta = 0.1$ and $N = 50$, after about 15×10^3 attempts we find the profile presented in figure 2 with $x_0/\delta = -0.78250438$, i.e. the last vortex is pushed out to the left from the pinning centre $m = 0$ almost completely. In this figure we also present the extreme profile obtained by Bryksin *et al* (1990a, b) where phase correlations at different pinning centres were neglected:

$$\begin{aligned} \alpha_{m+1}^2 &= \alpha_m^2 + (4/3\sqrt{3})(l/\delta)[(1+2\alpha_m^2)(1-\alpha_m^2)(1+\alpha_m^2) \\ &\quad + 2(1+\alpha_m^2+\alpha_m^4)^{3/2}]^{1/2} && \text{at } m > 0 \\ \alpha_m^2 &= 0 && \text{at } m \leqslant 0 \\ \delta n_m &= \sqrt{1+\alpha_m^2}/2K[1/(1+\alpha_m^2)]. \end{aligned} \tag{5}$$

Here $K(x)$ is the elliptic integral of the first type. In the region $H_{c1} \ll \bar{H} \ll H_a (H_{c1} \equiv 2\Phi_0/\pi^2\delta d$, $H_a \equiv \Phi_0/2ld$, $d = 2\lambda_L + d'$ is the junction effective thickness and λ_L is London penetration depth) the magnetic profile (5) takes the simple form (Bryksin *et al* 1989a, b)

$$\bar{H} = H_{c1}\pi l(x - y_0)/2\delta L \tag{6}$$

$$J_c = 2j_c l/\delta \tag{7}$$

which is equivalent to that obtained from the Bean (1962, 1964) theory. Here $j_c \equiv \hbar c^2/8\pi\delta^2 ed$ is the critical Josephson current and y_0 is a parameter that does not coincide with the front position.

Figure 2 shows that at fields $H_{c1} \ll \bar{H} \ll H_a$ the computed magnetic profile has a shape which is close to linear. Therefore the magnetic penetration has a Bean character.

The phase correlation may be neglected if the jump of the vortex density at every pinning centre is sufficiently large: $\Delta n_m L \simeq Ll/\delta^2 \gg 1$, where $\Delta n_m \equiv n_m - n_{m-1}$ (see (5) and Bryksin *et al* (1990a, b)). Therefore, the larger the L, the closer is the extreme profile to the profile presented by the broken line in figure 2. Nevertheless at a fixed L the extreme profile will lie below the line. Moreover it should be noted that for a finite number of attempts we cannot achieve the extreme profile. So the true extreme profile is placed between the crosses and the broken line in figure 2. This deviation of the computed profiles from the profile predicted by the local theory shows the absence of the local relation between the magnetic induction $B(x)$ and the critical current $J_c(B)$.

Now we consider the case of small L. For all initial conditions which we used we could not find non-periodic solutions at least for a length of order 100 L (see figure 3 for $L/\delta = l/\delta = 0.1$, curves B and C).

Let us discuss possible solutions which decrease at $x \to -\infty$. At small L/δ, equation (4) is useless for the initial condition at the 'front'. One must use the asymptotic relation between the phase ϑ and the derivative ϑ' obtained in appendix A on the basis of a linear approach:

$$\vartheta_0'/\vartheta_0^{(3)} = 1 - (2l/\delta)[\![2l/\delta - 1 + \exp(L/\delta)\{\cosh(L/\delta) + (l/\delta)\sinh(L/\delta)) + \sqrt{[\cosh(L/\delta) + (l/\delta)\sinh(L/\delta)]^2 - 1}\}]\!]^{-1}. \tag{8}$$

This equation ensures that ϑ and ϑ' tend to zero at $x \to -\infty$. We find that all solutions obeying the initial condition (8) at $x = 0$ are not different from the one-vortex solution, at least for a scale of $100L$ periods (see curve A in figure 3). The parameters of these solutions are renormalized and differ from parameters of the one-vortex solution for a uniform Josephson junction (Bryksin *et al* 1989b). Thus at small L/δ for the junction with pinning centres arranged periodically we cannot find 'chaotic solutions' of the type presented in figure 2. However, it should be noted that in terms of our computer procedure we consider the system of finite length (almost $100L$) only. Therefore we cannot state that no stochasticity appears for larger lengths in our system.

We can discuss two possibilities. Firstly, at all lengths, only periodical solutions are possible. This means that, at $H > H_{c1}^*$, vortices penetrate uniformly into the whole sample and no critical state exists (here H_{c1}^* is the effective critical field for the system considered). This situation is realized at $\Delta n_m L \simeq Ll/\delta^2 \leqslant 1$. The same parameter ($2Ll/\delta^2$) appears in the framework of the Frenkel–Kontorova model:

$$E = E_0 \sum_m \left(\frac{(\vartheta_{m+1} - \vartheta_m)^2}{2} + 2\lambda(1 - \cos\vartheta_m)\right). \tag{9}$$

In appendix B it is proved that at $L \ll l \ll \delta$ our model is equivalent to the Frenkel–Kontorova model with $\lambda = 2Ll/\delta^2$ and $E_0 = E_J\delta/16l$ where E_J is the Josephson energy. It is well known that the parameter λ determines the depinning threshold of incommensurate structures in the Frenkel–Kontorova model (Peyrard and Aubry 1983, de Seze and Aubry 1984).

Secondly, a perceptible change in the vortex density can occur on sufficiently large scales. This makes it difficult to observe a deviation from a regular solution. Let us estimate a minimum distance between two last vortices at the front of a magnetic field penetrating into the junction. As we have noted above, at $L \ll l \ll \delta$, the considered system may be described by the model (9) (see also appendix B). In terms of the Frenkel–Kontorova model the energy of a separated vortex in a multi-vortex solution depends on the coordinate x_0 of its centre:

$$E_v(x_0)/E_0 = 8\sqrt{\lambda} + 16\pi^2\exp(-\pi^2/\sqrt{\lambda})\cos(2\pi x_0/L). \tag{10}$$

The interaction energy of two vortices placed at a distance $\mathscr{L}$ from each other which is much greater than the soliton width δ^* is determined by (Frank and van der Merwe 1949)

$$E_{\rm int}(\mathscr{L})/E_0 = 32\sqrt{\lambda}\exp(-\mathscr{L}\sqrt{\lambda}/L) \qquad \mathscr{L} \gg \delta^* \equiv L\lambda^{-1/2}. \tag{11}$$

The minimum distance between these two vortices is determined by the equality of the repulsive force $\partial E_{\rm int}(\mathscr{L})/\partial\mathscr{L}$ to a pinning force $F_{\rm p}$:

$$F_{\rm p} = 32\pi^3(E_0/L)\exp(\pi^2/\sqrt{\lambda}) = 32\lambda(E_0/L)\exp(\sqrt{\lambda}\,\mathscr{L}/L). \tag{12}$$

Hence we obtain

$$\mathscr{L}_{\rm min}/L \sim \pi^2/\lambda \sim \pi^2\delta^2/2Ll. \tag{13}$$

Therefore in the considered range $L \ll l \ll \delta$ the distance (13) between vortices at the magnetic field front may be large enough. We think that the second variant is preferable, for in the Frenkel–Kontorova model only some incommensurate states are depinned (Aubry 1980, Peyrard and Aubry 1983).

Appendix A

Here we derive equation (8). Let us consider the behaviour of the phase ϑ in the region of the linear regime: $\vartheta, \vartheta'\delta \ll 1$. Let $\vartheta(x) = \vartheta'(x)$ at $x \to -\infty$. A solution of the linearized equation (1) in the interval between the centres m and $m+1$ may be written as

$$\vartheta(x) = a_m\exp[(x-mL)/\delta] + b_m\exp[-(x-mL)/\delta]. \tag{A1}$$

Substituting (A1) into (2) and (3) one obtains

$$\begin{aligned} a_{m-1}\exp(L/\delta) &= a_m(1-l/\delta) + b_m l/\delta \\ b_{m-1}\exp(-L/\delta) &= -a_m l/\delta + b_m(1+l/\delta). \end{aligned} \tag{A2}$$

We shall look for a_m and b_m in the form

$$a_m = \alpha t^{-m} \qquad b_m = \beta t^{-m}. \tag{A3}$$

Hence for unknown α and β we have

$$\begin{aligned} &\alpha[1-l/\delta - t\exp(L/\delta)] + \beta l/\delta = 0 \\ &-\alpha l/\delta + \beta[1+l/\delta - t\exp(-L/\delta)] = 0. \end{aligned} \tag{A4}$$

From the condition that the related matrix is equal to zero we obtain

$$t_{1,2} = \cosh(L/\delta) + (l/\delta)\sinh(L/\delta) \pm \{[\cosh(L/\delta) + (l/\delta)\sinh(L/\delta)]^2 - 1\}^{1/2} \tag{A5}$$

where $t_1 > 1$ and $t_2 < 1$. Hence

$$\begin{pmatrix} a_m \\ b_m \end{pmatrix} = \gamma\begin{pmatrix} 1 \\ c_1 \end{pmatrix} t_1^{-m} + \eta\begin{pmatrix} 1 \\ c_2 \end{pmatrix} t_2^{-m} \tag{A6}$$

where γ and η are arbitrary and

$$\begin{aligned} c_{1,2} = \beta_{1,2}/\alpha_{1,2} &= -[\delta - l - \delta t_{1,2}\exp(L/\delta)]/l \\ &= -(\delta/l)(1 - l/\delta - \exp(L/\delta)\,[\![\cosh(L/\delta) + (l/\delta\sinh(L/\delta) \end{aligned}$$

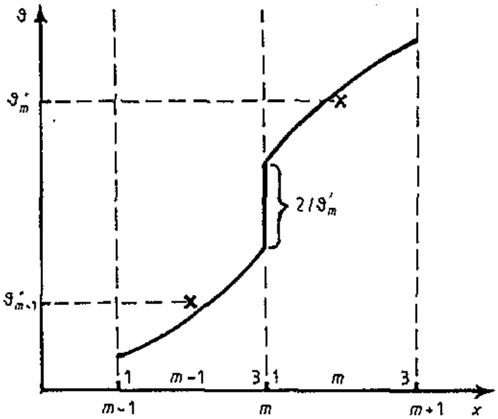

Figure B1. The phase difference ϑ versus x.

$$\pm \{[\cosh(L/\delta) + (l/\delta)\sinh(L/\delta)]^2 - 1\}^{1/2}]\!]). \tag{A7}$$

The parameters γ and η may be found from the condition that the solution under consideration decreases at $x \to -\infty$; so we obtain the desired relation (8).

Appendix B

Now we show that at $\lambda_L \ll L \ll l \ll \delta$ the Josephson junction with regularly arranged pinning centres with spacing L may be described by the Frenkel–Kontorova model. We shall use the approach developed by Bryksin *et al* (1989b) and the notation system is presented in figure B1.

Let us denote $\vartheta_m^+ \equiv (\vartheta_m^{(1)} + \vartheta_m^{(3)})/2$. Since the parameter

$$\alpha^2 \equiv \tfrac{1}{2}\{\tfrac{1}{2}[\delta\vartheta'(x)]^2 + \cos\vartheta(x) - 1\} \tag{B1}$$

is independent of x between neighbouring defects, we can easily see that

$$(2/\delta^2)(\cos\vartheta_m^{(1)} - \cos\vartheta_m^{(3)}) = (\vartheta'_{m+1})^2 - (\vartheta'_m)^2 \simeq 2\vartheta'_m(\vartheta'_{m+1} - \vartheta'_m). \tag{B2}$$

Substituting $\vartheta'_m \simeq (\vartheta_{m+1}^{(3)} - \vartheta_m^{(1)})/L$ into (B1) one obtains

$$\vartheta'_{m+1} - \vartheta'_m = (L/\delta^2)\sin\vartheta_m^+. \tag{B3}$$

On the other hand, taking into account the phase jump near pinning centres $\vartheta_m^{(1)} - \vartheta_m^{(3)} = 2l\vartheta'_m$, we obtain the following relation:

$$\vartheta'_m = [1/(L + 2l)](\vartheta_m^+ - \vartheta_{m-1}^+) \tag{B4}$$

(see figure B1). Substitution of (B4) into (B3) gives

$$(\delta^{*2}/L^2)(\vartheta_{m+1}^+ - 2\vartheta_m^+ + \vartheta_{m-1}^+) = \sin\vartheta_m^+. \tag{B5}$$

Here $\delta^* \equiv \delta(2l/L)^{-1/2}$ is the effective Josephson length (soliton width). It is evident that $\max(\vartheta') = 2/\delta^*$. Taking into account the transverse junctions, and using (B5) and the expression for the free energy of a Josephson junction, we obtain the related energy functional:

$$\mathcal{F} = \frac{\hbar}{2e} j_c L \sum_m \left(1 - \cos\vartheta_m^+ + \frac{\delta^{*2}}{2^{L2}}(\vartheta_m^+ - \vartheta_{m-1}^+)^2\right) \tag{B6}$$

which coincides with the Frenkel–Kontorova model functional. Then the soliton energy is

$$E/E_J = H_{c1}^*/H_{c1} = (2l/L)^{-1/2}. \tag{B7}$$

Thus the solution pinning energy is

$$E_p = 4\pi E_J \exp[-(\pi^2\delta/2L)\sqrt{L/2l}] \tag{B8}$$

where $E_J \equiv (4\hbar/e)j_c\delta$ is the Josephson energy and $H_{c1} \equiv 4\pi E_J/\Phi_0$.

For an arbitrary relation between l and δ, one can use the approximate equation

$$\delta^* = \delta[(2\delta/L)\tanh(l/\delta)]^{-1/2} \tag{B9}$$

which at $\delta \ll l$ corresponds to the results of Bryksin *et al* (1989b).

So the parameters of Frenkel–Kontorova model, used in equations (9)–(13), are

$$\lambda = 2lL/\delta^2 \tag{B10}$$

$$E_0 = (\hbar/2e)j_c(\delta^2/l) = (\delta/16l)E_J. \tag{B11}$$

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